1.(6pts) Find $\mathrm{d}z/\mathrm{d}t$ when $t = 0$, where $z = x^2 + y^2 + 2xy$, $x = \ln(t + 1)$ and $y = e^{3t}$.

(a) 8 (b) 2 (c) 1 (d) 6 (e) 5

Solution. Notice that $x(0) = \ln(1) = 0$ and $y(0) = 1$. By the chain rule

$$\begin{aligned} \left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} &= \left.\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} \\ &= (2x + 2y)\left(\frac{1}{t+1}\right) + (2x + 2y)3e^{3t}\Big|_{t=0} \\ &= 2(1) + (2)(3) = 8. \end{aligned}$$

2.(6pts) Calculate the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at the point $(2, 4, 2)$ in the direction of the vector $\langle 1, 2, 1\rangle$.

(a) $\frac{24}{\sqrt{6}}$ (b) $\sqrt{6}$ (c) $-\frac{1}{12}\langle 1, 4, 1\rangle$ (d) -9.79 (e) $\left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

Solution.

$$\nabla f(2, 4, 2) = \langle 2x, 2y, 2z\rangle_{(2,4,2)} = \langle 4, 8, 4\rangle$$

So

$$D_{\langle 1,2,1\rangle} f(2, 4, 2) = \langle 4, 8, 4\rangle \bullet \langle 1, 2, 1\rangle \frac{1}{\sqrt{6}} = \frac{4 + 16 + 4}{\sqrt{6}} = \frac{24}{\sqrt{6}}$$

3.(6pts) What is the normal line to $z^2 = 9x^2 - 4y^2$ at $(2, 3, 0)$?

(a) $\langle 2, 3, 0\rangle + t\langle 36, -24, 0\rangle$ (b) $\langle 2, 3, 0\rangle + t\langle 18x, -8y, -2z\rangle$

(c) $\langle 2, 3\rangle + t\langle 36, -24\rangle$ (d) $36x - 24y = 0$

(e) $\langle 2, 3, 0\rangle + t\langle 24, 36, -12\rangle$

Solution. Rewrite the surface as $f(x, y, z) = 9x^2 - 4y^2 - z^2 = 0$. Then $\nabla f = \langle 18x, -8y, -2z\rangle$, which is $\langle 36, -24, 0\rangle$ at the point $(2, 3, 0)$. The gradient is normal to the surface, so it gives the direction of the normal line. The equation is therefore $\langle 2, 3, 0\rangle + t\langle 36, -24, 0\rangle$.

4.(6pts) Find and classify the critical points of $f(x, y) = x^2 + 6xy - 4y^2$.

(a) $(0, 0)$, saddle point. (b) $(0, 0)$, local maximum.

(c) $(0, 0)$, local minimum. (d) $(1, 1)$, saddle point.

(e) $(1, 1)$, local maximum.

Solution. The first and second partial derivatives are given by $f_x = 2x + 6y$, $f_y = 6x - 6y$, $f_{xx} = 2$, $f_{yy} = -6$, $f_{xy} = 6$. From the first partial derivatives we see that the only critical point is at $(0, 0)$. The Hessian is also constant and negative, hence this critical point is a saddle point.

5.(6pts) Suppose $f(x, y) = x^2y$ with domain $D = \{(x, y)|x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$. What is the absolute maximum value of $f(x, y)$?

(a) 2 (b) 1 (c) 3 (d) 4 (e) 5

Solution. Interior: If (x, y) is a critical point, then $f_y(x, y) = x^2 = 0$ and (x, y) is not an interior point.

Boundary: If $x = 0$ or $y = 0$, then $f(x, y) = 0$.

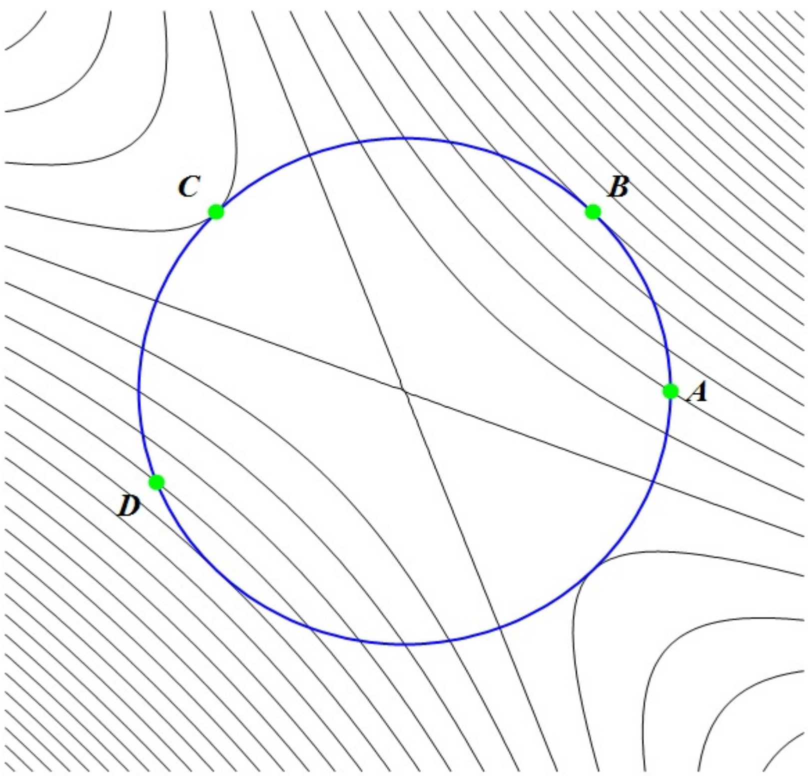
Otherwise, (x, y) is on the curved part of the boundary where $x^2 = 3 - y^2$ and $f(x, y) = g(y) = 3y - y^3$ with $0 \le y \le \sqrt{3}$. Since $g'(y) = 3 - 3y^2$ we get that $y = 1$ is a critical point for this problem. Then $x = \sqrt{2}$ and $f(\sqrt{2}, 1) = 2$.

Since the region is closed and bounded, f must have a maximum value and since $2 > 0$, 2 is it.

OR

Lagrange multipliers: $\left\langle 2xy, x^2 \right\rangle = \lambda \left\langle 2x, 2y \right\rangle$. One solution is $x = 0$, hence $y = \sqrt{3}$, $\lambda = 0$. Here $f = 0$. If $x \neq 0$, $\lambda = y$, $2y^2 = x^2$ and $y = \dfrac{x}{\sqrt{2}}$ (the other solutions are not in the region). Hence $x^2 + \dfrac{x^2}{2} = 3$ or $\dfrac{3x^2}{2} = 3$ so $x^2 = 2$ and at this point $f = 2$ and as above this must be the maximum value.

6.(6pts) Consider the following contour plot for a function $f(x, y)$:



The circle is a level curve $g(x, y) = k$. Which of the following must ALWAYS be true?

(a) Subject to $g(x, y) = k$, $f(x, y)$ has a possible extremum at C.

(b) Subject to $g(x, y) = k$, $f(x, y)$ has a possible maximum at A.

(c) Subject to $g(x, y) = k$, $f(x, y)$ has a possible minimum at D.

(d) Subject to $g(x, y) = k$, $f(x, y)$ has an absolute maximum at B.

(e) $f(x, y)$ has a possible absolute maximum or absolute minimum at C.

Solution. At A and at D, ∇f is not parallel to ∇g, so neither A or D can be an extremum of f subject to $g = k$. B is a potential extremum of f subject to $g = k$, but it could be that B is a absolute minimum, or just a local minimum/maximum. The statement "$f(x, y)$ has a possible absolute maximum or minimum at C" is wrong since the gradient of f at C is not zero.

On another note, it is worthwhile to note that the Lagrange multipliers theorem says nothing about the extrema of f itself, but of f restricted to $g = k$.

Thus $f(x, y)$, subject to $g = k$, having a possible extremum at C is the correct answer since here ∇f is parallel to ∇g and $\nabla g \neq 0$ at C, so this satisfies the hypothesis of the Lagrange multiplier theorem naming C as a candidate extremum point.

7.(6pts) Evaluate the following double integral

$$\iint_R (5 - x)\, dA$$

for $R = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$.

(a) 36 (b) 24 (c) 60 (d) 12 (e) 52

Solution. The region R is a rectangle in the xy-plane, and since $z = 5 - x$ we see that we are computing the volume of a solid which can be viewed as a rectangular prism with base R and height 1, together with a (right) triangular prism on top of the rectangular solid with base R and height 4. So we can compute the integral by computing the volumes of the two solids and adding.

Volume of the rectangular prism: $V = l * w * h = 4 * 3 * 1 = 12$

Volume of the triangular prism: $V = \frac{1}{2} l * w * h = \frac{1}{2} 4 * 3 * 4 = 24$

So the total volume is 36.

OR

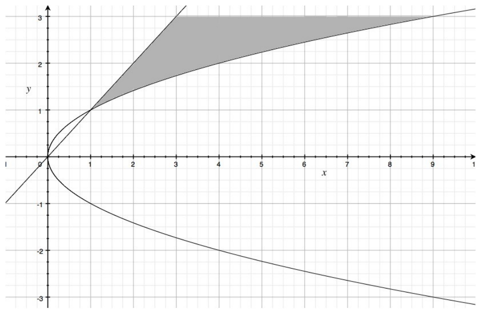
$\iint_R (5 - x)\, dA = \int_0^4 \int_0^3 (5 - x)\, dy\, dx = \int_0^4 (5 - x)\Big|_{y=0}^{y=3} dx = \int_0^4 (15 - 3x)\, dx = 15x - \frac{3x^2}{2}\Big|_{x=0}^{x=4} = 60 - 24 = 36.$

OR

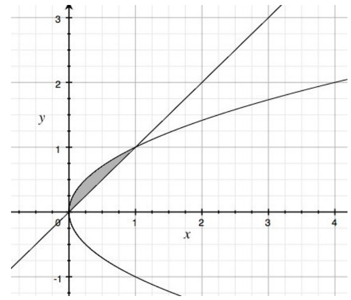
$\iint_R (5 - x)\, dA = \int_0^3 \int_0^4 (5 - x)\, dx\, dy = \int_0^3 \left(5x - \frac{x^2}{2}\right)\Big|_{x=0}^{x=4} dy = \int_0^3 (20 - 8)\, dy = \int_0^3 12\, dy = 12y\Big|_{y=0}^{y=3} = 36.$

8.(6pts) Consider the double integral of a function f over a region R, $\iint_R f\,dA$. Suppose $\iint_R f\,dA = \int_1^3 \int_y^{y^2} f(x,y)\,dx\,dy$. Which gray region below is R?

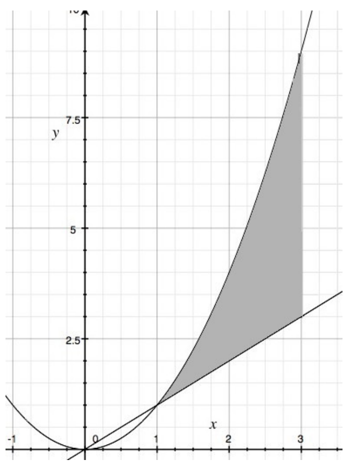
(a)



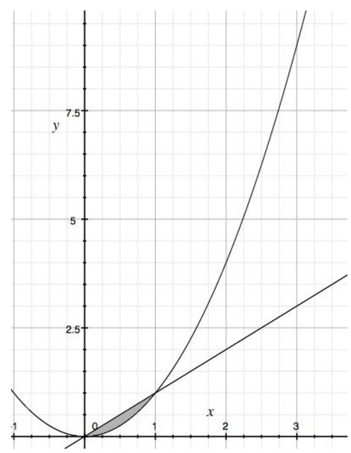
(b)



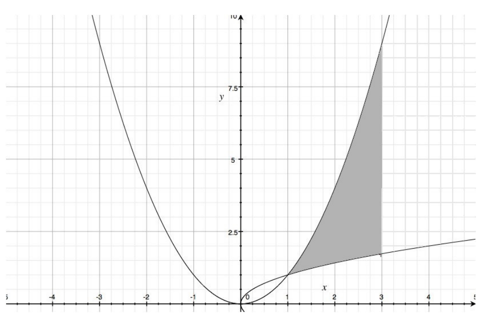
(c)



(d)



(e)



Solution. The region R can be described in formulas as the set of all (x,y) such that $1 \leqslant y \leqslant 3$ and $y \leqslant x \leqslant y^2$. There is only one region which lies between $y = 1$ and $y = 3$.

9.(10pts) (a) Find an equation for the tangent line (in vector or parametric form) at the point $(2, 2, 1)$ to the curve of intersection of the two surfaces $g(x, y, z) = 2x^2 + 2y^2 + z^2 = 17$ and $h(x, y, z) = x^2 + y^2 - 3z^2 = 5$. (8 pts)

(b) Suppose $f(x, y, z)$ is a function with $\nabla f = \langle 0, 1, 0 \rangle$ at the point $(2, 2, 1)$. Starting at $(2, 2, 1)$, which direction should one travel along the curve of intersection in order to increase f? (2 pts)

Note: You can specify a direction along the curve by saying whether the variable in your equation from (a) would increase or decrease, or by choosing a vector tangent to the curve.

Solution. (a) The line is in the tangent plane to each surface, so its direction is perpendicular to both normal vectors. The normal vectors are $\nabla g = \langle 4x, 4y, 2z \rangle = \langle 8, 8, 2 \rangle$ and $\nabla h = \langle 2x, 2y, -6z \rangle =< 4, 4, -6 >$. The cross product $\nabla g \times \nabla h = \langle -56, 56, 0 \rangle$ will serve as a direction vector. $\langle 2, 2, 1 \rangle + t \langle -56, 56, 0 \rangle$ is an equation for the tangent line.

(b) Let $\mathbf{u}$ be a unit vector which points in the same direction as $\langle -56, 56, 0 \rangle$. Since $D_{\mathbf{u}} f = \dfrac{\langle 0, 1, 0 \rangle \bullet \langle -56, 56, 0 \rangle}{56\sqrt{2}} = \dfrac{1}{\sqrt{2}} > 0$ at $(2, 2, 1)$, one should increase t in order to increase f.

10.(10pts) Find the absolute maximum and minimum of $f(x, y, z) = 2x + y$ with respect to the constraints $g(x, y, z) = 2x^2 + z^2 = 4$ and $h(x, y, z) = 2x + y + 3z = 6$.

Solution.

$$\nabla f = \langle 2, 1, 0 \rangle$$
$$\nabla g = \langle 4x, 0, 2z \rangle$$
$$\nabla h = \langle 2, 1, 3 \rangle$$

So

$$2 = 4x\lambda + 2\mu$$
$$1 = \mu$$
$$0 = 2\lambda + 3\mu$$

Using the second equation on the first equation we get

$$2 = 4x\lambda + 2$$

This reduces to

$$0 = 4x\lambda$$

This implies $x = 0$ or $\lambda = 0$. Let try $\lambda = 0$ in the third equation above. That yields $0 = 3$ which is a contradiction. So $x = 0$. Now we can use our restraints to find y, z. Using $g(x, y, z) = 2x^2 + z^2 = 4$, we get $z^2 = 4$ or $z = \pm 2$. Using $h(x, y, z) = 2x + y + 3z = 6$ we see that when $x = 0$ and $z = 2$ that $y + 6 = 6$ so that $y = 0$ and that when $x = 0$ and $z = -2$ that $y - 6 = 6$ so that $y = 12$. So our critical points are $(0, 0, 2)$ and $(0, 12, 2)$. $f(0, 0, 2) = 0$ for an absolute minimum and $f(0, 12, -2) = 12$ for an absolute maximum.

11.(10pts) Find and classify all critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Solution. Begin by finding all first and second partial derivatives: $f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$, $f_{xx} = 6y - 6$, $f_{xy} = 6x$, $f_{yy} = 6y - 6$. We now need the critical points. Find these by solving the equations

$$f_x = 6xy - 6x = 0$$
$$f_y = 3x^2 + 3y^2 - 6y = 0$$

The first equation factors as $6x(y-1) = 0$ so it will be zero if $x = 0$ or $y = 1$. The most common mistake here was to forget the $x = 0$ solution. To find the critical points we can plug these values into f_y and solve for the remaining variable. For $x = 0$ we have $f_y = 3y^2 - 6y = 0$ which implies $y = 0$ or $y = 2$. For $y = 1$ we have $f_y = 3x^2 - 3 = 0$ which implies $x = 1$ or $x = -1$. So if $x = 0$ we have the critical points $(0,0)$ and $(0,2)$. If $y = 1$ we have the critical points $(1,1)$ and $(-1,1)$. Now all we need to do is classify the critical points. The discriminant $D(x,y)$ is given by

$$D(x,y) = (6y-6)^2 - 36x^2$$

$(0,0)$: $D(0,0) = 36 > 0$ and $f_{xx}(0,0) = -6 < 0$. $(0,2)$: $D(0,2) = 36 > 0$ and $f_{xx}(0,2) = 6 > 0$. $(1,1)$: $D(1,1) = -36 < 0$. $(-1,1)$: $D(-1,1) = -36 < 0$. So $(0,0)$ is a relative max, $(0,2)$ is a relative min, and $(1,1), (-1,1)$ are saddle points.

12.(10pts) A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is changing at a rate of -3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

Solution. We have $V = \pi r^2 \ell$, where V is the volume, r the radius and ℓ the length, and each of r and ℓ are functions of the time, t. Since the fluid is incompressible $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 0$.

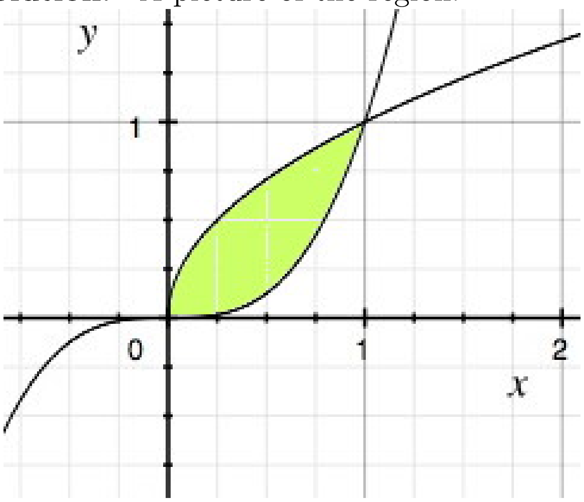
By the chain rule, this is

$$0 = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial \ell}\frac{\mathrm{d}\ell}{\mathrm{d}t} = 2\pi r\ell\frac{\mathrm{d}r}{\mathrm{d}t} + \pi r^2\frac{\mathrm{d}\ell}{\mathrm{d}t}.$$

Filling in $\mathrm{d}\ell/\mathrm{d}t = -3$ we obtain $\mathrm{d}r/\mathrm{d}t = 3\pi r^2/2\pi r\ell = 3r/2\ell$. When $r = 2$, $\ell = 1$ this gives $\mathrm{d}r/\mathrm{d}t = 3$m/s.

13.(10pts) Evaluate $\displaystyle\iint_R 4xy\,dA$ where R is the region bounded above by $y = \sqrt{x}$ and below by $y = x^3$.

Solution. A picture of the region:



To set up the double integral as an iterated integral $dy\,dx$ we first need bounds for x. Clearly we start when $x = 0$ and end when $x = 1$ or more formally we need to find when $x^3 = \sqrt{x}$ or $x^6 = x$ or $x = 0$, $x^5 = 1$, which has solutions $x = 0$, 1. Then the limits on the inner integral are x^3 at the bottom and $\sqrt{x}$ at the top.

$$\iint_R 4xy\,dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx = \int_0^1 2xy^2\Big|_{x^3}^{\sqrt{x}} dx = \int_0^1 2x^2 - 2x^7\,dx = \frac{2x^3}{3} - \frac{2x^8}{8}\Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Or we could set up the double integral as an iterated integral $dx\,dy$. This time we need to know the y-coordinates of the intersection points but the same algebra as above gives $y = 0$, 1. The limits on the inner integral start at the right hand curve whose x-coordinate in terms of y is $x = y^2$. The upper limit is the x-coordinate of the right-han curve in terms of y which is $x = \sqrt[3]{y}$. Hence

$$\iint_R 4xy\,dA = \int_0^1 \int_{y^2}^{\sqrt[3]{y}} 4xy\,dx\,dy = \int_0^1 2x^2y\Big|_{x=y^2}^{x=\sqrt[3]{y}} dx = \int_0^1 2y^{5/3} - 2y^5\,dy = 2\frac{3}{8}y^{8/3} - \frac{2y^6}{6}\Big|_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}.$$